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The construction of exact rational solutions with constant asymptotic values at infinity of two-dimensional NVN integrable nonlinear evolution equations via the $\overline{\partial}$-dressing method

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Abstract

The classes of exact rational solutions with constant asymptotic values at infinity of Nizhnik–Veselov–Novikov (NVN) equations via the $\overline{\partial}$-dressing method of Zakharov and Manakov are constructed. At fixed time such solutions are the transparent (exactly solvable) potentials for one-dimensional Klein–Gordon or two-dimensional stationary Schrödinger equations. Among the constructed solutions are singular and also non-singular ones.

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1. Introduction

Over the past two decades the inverse spectral transform (IST) method has been generalized and successfully applied for the calculations of broad classes of exact solutions of various (2 + 1)-dimensional nonlinear evolution equations such as Kadomtsev–Petvashvili, Davey–Stewartson, Veselov–Novikov, Zakharov–Manakov system, generalized sine–Gordon and others (see the books [1–4] and references therein). The basic tools for solving (2 + 1)-dimensional integrable nonlinear equations via IST are now the non-local Riemann–Hilbert problem [5], the $\overline{\partial}$-problem [6] and the more general and powerful $\overline{\partial}$-dressing method of Zakharov and Manakov [7–10] (see also the reviews [11–13] and the books [1–4]).

In the present paper the $\overline{\partial}$-dressing method is used to construct exact rational solutions with constant asymptotic values at infinity of the famous (2 + 1)-dimensional Nizhnik–Veselov–Novicov (NVN) integrable equations:

$$U_t + \kappa_1 U_{\xi\xi\xi} + \kappa_2 U_{\eta\eta\eta} + 3\kappa_1 (U\partial_\xi^{-1} U)_\eta + 3\kappa_2 (U\partial_\eta^{-1} U_\xi)_\xi = 0 \tag{1}$$

where $U(\xi, \eta, t)$ is a scalar function, κ_1, κ_2 are arbitrary constants, $\partial_\xi = \partial_x + \sigma\partial_y$, $\partial_\eta = \partial_x - \sigma\partial_y$ and $\sigma^2 = \pm 1$. Equation (1) was first introduced by Nizhnik [14] for $\sigma = 1$ and independently

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by Veselov and Novikov [15] for $\sigma = \mathrm{i}, \kappa_1 = \kappa_2 = 1$. Here and below $\partial_\xi^{-1}, \partial_\eta^{-1}$ denote operators inverse to $\partial_\xi, \partial_\eta$: $\partial_\xi^{-1}\partial_\xi = \partial_\eta^{-1}\partial_\eta = 1$. The integrability of (1) by IST and by another means is based on the representation of this equation as the compatibility condition for two linear auxiliary problems

$$
\begin{aligned}
L_1\psi &= (\partial^2_{\xi\eta} + U)\psi = 0 \\
L_2\psi &= (\partial_t + \kappa_1\partial_\xi^3 + \kappa_2\partial_\eta^3 + 3\kappa_1(\partial_\xi^{-1}U_\eta)\partial_\xi + 3\kappa_2(\partial_\eta^{-1}U_\xi)\partial_\eta)\psi = 0
\end{aligned}
\tag{2}
$$

in the form of Manakov's triad

$$
[L_1, L_2] = BL_1 \tag{3}
$$

with

$$
B = 3(\kappa_1\partial_\xi^{-1}U_{\eta\eta} + \kappa_2\partial_\eta^{-1}U_{\xi\xi}). \tag{4}
$$

The integration of the NVN equation (1) has a remarkable history. In the paper by Nizhnik [14], equation (1) with $\sigma = 1$ was integrated using the inverse problem technique for hyperbolic systems on the plane. In the paper by Veselov and Novikov [15] for the construction of the periodic finite-zone exact solutions of (1) with $\sigma = \mathrm{i}$ algebraic geometric methods were used. Finally, in the papers by Grinevich and Manakov [16], Grinevich and Novikov [17, 18] for the integration of (1) with $\sigma = \mathrm{i}$ and for potentials which decay quickly at infinity, an IST based on a combination of some non-local Riemann–Hilbert problem and the $\bar\partial$-problem has been developed. More detailed information about all known cases of exact integrations of (1) (early history) can be found in the book [3]. Let us also mention that for the calculation of exact solutions of (1) the methods of Darbu and Bäcklund transformations have been used [19].

In our paper the $\bar\partial$-method corresponds to bare operators of linear auxiliary problems (2) with a constant asymptotic value of U at infinity,

$$
U(\xi,\eta,t) := \tilde{U}(\xi,\eta,t) - \epsilon \qquad U(\xi,\eta,t) \xrightarrow[x^2+y^2\longrightarrow\infty]{} -\epsilon \neq 0. \tag{5}
$$

In this case the first linear auxiliary problem (1.2) has the form

$$
(\partial^2_{\xi\eta} + \widetilde{U})\psi = \epsilon\psi. \tag{6}
$$

For $\sigma = 1$ (6) can be interpreted ($\xi \Rightarrow t - x, \eta \Rightarrow t + y$) as the one-dimensional Klein–Gordon or perturbed telegraph equation, for $\sigma = \mathrm{i}$ (6) is nothing but the two-dimensional (2D) stationary Schrödinger equation. The construction of exact solutions of (1) with constant asymptotic values at infinity means simultaneously the calculation of exact eigenfunctions (or wavefunctions, in the terminology of quantum mechanics) ψ and exactly solvable (transparent) corresponding potentials $\widetilde{U}$ for the above-mentioned classical linear equations.

In our opinion, the use of the celebrated $\bar\partial$-method of Zakharov and Manakov for the construction of new exact solutions for NVN equations (1) is very instructive and useful. In the case $\sigma = \mathrm{i}$ our results partially agree with those obtained by different methods in the papers by Grinevich and Novikov [17, 18]. We shall calculate the solutions corresponding to the simple poles of the wavefunction. The study of multiple-pole solutions for the NVN equation is in progress and will be considered elsewhere.

The paper is organized as follows. In section 2 the basic ingredients of the $\bar\partial$-dressing method for the NVN equation (1) are considered. The general formulae for rational solutions corresponding to a factorized delta-kernel R of the $\bar\partial$-problem are obtained at the end of section 2. The rational solutions of NVN equations (1) in the Nizhnik case ($\sigma = 1$) and the Veselov–Novikov case ($\sigma = \mathrm{i}$) are calculated in sections 3 and 4, respectively.

2. Basic formulae of the $\overline{\partial}$-dressing method

Let us apply the $\overline{\partial}$-dressing method [7–10] for equation (1) in the case when $U(\xi, \eta, t)$ has generically a non-zero asymptotic value at infinity:

$$U(\xi, \eta, t) = \widetilde{U}(\xi, \eta, t) + U_\infty = \widetilde{U}(\xi, \eta, t) - \epsilon \tag{7}$$

where $\widetilde{U}(\xi, \eta, t) \longrightarrow 0$ as $\xi^2 + \eta^2 \rightarrow \infty$. At first one postulates the non-local $\overline{\partial}$-problem [7, 13]:

$$\frac{\partial \chi}{\partial \overline{\lambda}} = (\chi * R)(\lambda, \overline{\lambda}) = \iint \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}\; \chi(\mu, \overline{\mu})\, R(\mu, \overline{\mu}; \lambda, \overline{\lambda}) \tag{8}$$

where in our case χ and R are the scalar complex-valued functions. For the function χ we choose the canonical normalization: $\chi \longrightarrow 1$ as $\lambda \rightarrow \infty$. We also assume that the problem (8) is uniquely solvable.

Then one introduces the dependence of kernel R of the $\overline{\partial}$-problem (8) on the space and time variables ξ, η, t [7, 13]:

$$\begin{aligned}
\frac{\partial R}{\partial \xi} &= \mathrm{i}\mu R(\mu, \overline{\mu}; \lambda, \overline{\lambda}; \xi, \eta, t) - R(\mu, \overline{\mu}; \lambda, \overline{\lambda}; \xi, \eta, t)\, \mathrm{i}\lambda \\
\frac{\partial R}{\partial \eta} &= -\mathrm{i}\frac{\epsilon}{\mu} R(\mu, \overline{\mu}; \lambda, \overline{\lambda}; \xi, \eta, t) + R(\mu, \overline{\mu}; \lambda, \overline{\lambda}; \xi, \eta, t)\, \mathrm{i}\frac{\epsilon}{\lambda} \\
\frac{\partial R}{\partial t} &= \mathrm{i}\left(\kappa_1 \mu^3 - \kappa_2 \frac{\epsilon^3}{\mu^3}\right) R(\mu, \overline{\mu}; \lambda, \overline{\lambda}; \xi, \eta, t) - R(\mu, \overline{\mu}; \lambda, \overline{\lambda}; \xi, \eta, t)\, \mathrm{i}\left(\kappa_1 \lambda^3 - \kappa_2 \frac{\epsilon^3}{\lambda^3}\right)
\end{aligned} \tag{9}$$

i.e.

$$R(\mu, \overline{\mu}; \lambda, \overline{\lambda}; \xi, \eta, t) = R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})\, \mathrm{e}^{F(\mu) - F(\lambda)} \tag{10}$$

where

$$F(\lambda) := \mathrm{i}\left(\lambda \xi - \frac{\epsilon}{\lambda}\eta\right) - \mathrm{i}\left(\kappa_1 \lambda^3 - \kappa_2 \frac{\epsilon^3}{\lambda^3}\right) t. \tag{11}$$

With the use of 'long' derivatives

$$D_1 = \partial_\xi + \mathrm{i}\lambda \qquad D_2 = \partial_\eta - \mathrm{i}\frac{\epsilon}{\lambda} \qquad D_3 = \partial_t + \mathrm{i}\left(\kappa_1 \lambda^3 - \kappa_2 \frac{\epsilon^3}{\lambda^3}\right) \tag{12}$$

the dependence (9) of kernel R of the $\overline{\partial}$-problem (8) on ξ, η, t can be expressed in the form:

$$[D_1, R] = 0 \qquad [D_2, R] = 0 \qquad [D_3, R] = 0. \tag{13}$$

With the use of derivatives (12) one can then construct the linear operators

$$L = \sum_{lmn} U_{lmn}(\xi, \eta, t)\, D_1^l D_2^m D_3^n \tag{14}$$

which satisfy the condition

$$\left[\frac{\partial}{\partial \overline{\lambda}}, L\right] = 0 \tag{15}$$

for absence of singularities on λ. For such operators L the function $L\chi$ obeys the same $\overline{\partial}$-equation as the function χ. If there are several operators L_i of this type, then by virtue of the unique solvability of (8) one has $L_i \chi = 0$. In our case one can construct two such operators:

$$\begin{aligned}
L_1 \chi &= (D_1 D_2 + V_1 D_1 + V_2 D_2 + U)\chi = 0 \\
L_2 \chi &= (D_3 + \kappa_1 D_1^3 + \kappa_2 D_2^3 + W_1 D_1^2 + W_2 D_2^2 + W_3 D_1 + W_4 D_2 + W)\chi = 0.
\end{aligned} \tag{16}$$

Indeed, let us consider (16) for the series expansions of χ near the points $\lambda = 0$ and

$$\lambda = \infty: \qquad \chi = \widetilde{\chi}_0 + \lambda\chi_1 + \cdots \qquad \chi = \chi_0 + \chi_{-1}/\lambda + \cdots .$$

In the neighbourhood of $\lambda = \infty$, equating to zero the coefficients for degrees of λ, we obtain from $L_1\chi = 0$

$$\begin{aligned}
\lambda: &\quad \mathrm{i}\chi_{0\eta} + \mathrm{i}V_1\chi_0 = 0\\
\lambda^0: &\quad \chi_{0\xi\eta} + \mathrm{i}\chi_{-1\eta} + \epsilon\chi_0 + V_1\chi_{0\xi} + V_2\chi_{0\eta} + U\chi_0 = 0\\
\lambda^{-1}: &\quad \chi_{-1\xi\eta} - \mathrm{i}\epsilon\chi_{0\xi} + \mathrm{i}\chi_{-2\eta} + \epsilon\chi_{-1} + V_2\chi_{-1\eta} - \mathrm{i}\epsilon V_2\chi_0 + U\chi_{-1} = 0
\end{aligned} \tag{17}$$

and from $L_2\chi = 0$

$$\begin{aligned}
\lambda^2: &\quad -3\kappa_1\chi_{0\xi} - W_1\chi_0 = 0\\
\lambda^1: &\quad 3\mathrm{i}\kappa_1\chi_{0\xi\xi} - 3\kappa_1\chi_{-1\xi} + 2\mathrm{i}W_1\chi_{0\xi} - W_1\chi_{-1} + \mathrm{i}W_4\chi_0 = 0\\
\lambda^0: &\quad \chi_{0t} + \kappa_1\chi_{0\xi\xi\xi} + 3\mathrm{i}\kappa_1\chi_{-1\xi\xi} - 3\kappa_1\chi_{-2\xi} + \kappa_2\chi_{0\eta\eta\eta} + W_1\chi_{0\xi\xi} + 2\mathrm{i}W_1\chi_{-1\xi}\\
&\qquad - W_1\chi_{-2} + W_3\chi_{0\eta} + W_4\chi_{0\xi} + \mathrm{i}W_4\chi_{-1} + W\chi_0 = 0.
\end{aligned} \tag{18}$$

Analogously, in the neighbourhood of $\lambda = 0$ from $L_1\chi = 0$

$$\begin{aligned}
\lambda^{-1}: &\quad -\mathrm{i}\epsilon\widetilde{\chi}_{0\xi} - \mathrm{i}V_2\epsilon\widetilde{\chi}_0 = 0\\
\lambda^0: &\quad \widetilde{\chi}_{0\xi\eta} - \mathrm{i}\epsilon\chi_{1\xi} + \epsilon\widetilde{\chi}_0 + V_2\widetilde{\chi}_{0\eta} - \mathrm{i}\epsilon V_2\chi_1 + U\widetilde{\chi}_0 = 0
\end{aligned} \tag{19}$$

and from $L_2\chi = 0$

$$\begin{aligned}
\lambda^{-2}: &\quad -3\kappa_2\epsilon^2\widetilde{\chi}_{0\eta} - \epsilon^2W_2\widetilde{\chi}_0 = 0\\
\lambda^{-1}: &\quad -3\mathrm{i}\kappa_2\epsilon\widetilde{\chi}_{0\eta\eta} - 3\kappa_2\epsilon^2\chi_{1\eta} - 2\mathrm{i}\epsilon W_2\widetilde{\chi}_{0\eta} - \epsilon^2W_2\chi_1 - \mathrm{i}\epsilon W_3\widetilde{\chi}_0 = 0.
\end{aligned} \tag{20}$$

Due to canonical normalization $\chi_0 = 1$ and from (17) and (18) it follows for V_1 and W_1: $V_1 = W_1 = 0$. Then from (19) and (20) we obtain for V_2 and W_2 the following reconstruction formulae:

$$V_2 = -\frac{\widetilde{\chi}_{0\xi}}{\widetilde{\chi}_0} \qquad W_2 = -3\kappa_2\frac{\widetilde{\chi}_{0\eta}}{\widetilde{\chi}_0}. \tag{21}$$

Imposing on the operator L_1 in (16) the condition of potentiality $V_2 = 0$ satisfying for $\widetilde{\chi}_0 =$ constant, say $\widetilde{\chi}_0 = 1$, we have from (21)

$$V_2 = W_2 = 0. \tag{22}$$

Let us mention that in the well known terminology the operator L_1 in (16) is a pure potential operator if the terms with the first derivatives in it are absent. For this reason the condition $V_2 = 0$ or equivalently $\tilde{\chi}_0 =$ constant will be called here the condition of potentiality of operator L_1.

Then from (17)–(22) one obtains the reconstruction formulae for U, W_3 and W_4:

$$\begin{aligned}
&U = -\epsilon - \mathrm{i}\chi_{-1\eta} = -\epsilon + \mathrm{i}\epsilon\chi_{1\xi}\\
&W_4 = -3\mathrm{i}\kappa_1\chi_{-1\xi} = 3\kappa_1\partial_\eta^{-1}U_\xi \qquad W_3 = 3\mathrm{i}\kappa_2\epsilon\chi_{1\eta} = 3\kappa_2\partial_\xi^{-1}U_\eta.
\end{aligned} \tag{23}$$

And finally using the last relations from (17), (18) and (23) one obtains the expression for W:

$$W = -3\mathrm{i}\kappa_1\chi_{-1\xi\xi} + 3\kappa_1\chi_{-2\xi} - \mathrm{i}W_4\chi_{-1} \equiv 0. \tag{24}$$

In terms of the wavefunction

$$\psi := \chi \exp\left[\mathrm{i}\left(\lambda\xi - \frac{\epsilon}{\lambda}\eta\right) - \mathrm{i}\left(\kappa_1\lambda^3 - \kappa_2\frac{\epsilon^3}{\lambda^3}\right)t\right] \tag{25}$$

under the reduction $V_2 = 0$ (the condition of potentiality L_1) one obtains from (16) due to (22)–(25) the linear auxiliary system

$$\begin{aligned} L_1\psi &= (\partial^2_{\xi\eta} + \widetilde{U} - \epsilon)\psi = (\partial^2_{\xi\eta} + U)\psi = 0 \\ L_2\psi &= (\partial_t + \kappa_1\partial^3_\xi + \kappa_2\partial^3_\eta + 3\kappa_1(\partial_\eta^{-1}U_\xi)\partial_\eta + 3\kappa_2(\partial_\xi^{-1}U_\eta)\partial_\xi)\psi = 0 \end{aligned} \tag{26}$$

which exactly coincides with the system (2). The compatibility conditions (3) for the system (26) is nothing but the NVN equation (1) for the function U with constant asymptotic value $-\epsilon$ at infinity.

The solution of the $\overline{\partial}$-problem (8) with canonical normalization $\chi_0 = 1$ is equivalent to the solution of the following singular integral equation:

$$\chi(\lambda) = 1 + \iint_C \frac{\mathrm{d}\lambda' \wedge \mathrm{d}\overline{\lambda'}}{2\pi\mathrm{i}(\lambda' - \lambda)} \iint_C \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}\ \chi(\mu, \overline{\mu})\, R_0(\mu, \overline{\mu}; \lambda', \overline{\lambda'})\, \mathrm{e}^{F(\mu)-F(\lambda')}. \tag{27}$$

From (27) one obtains for the coefficients $\widetilde{\chi}_0$ and χ_{-1} of the series expansions of χ:

$$\widetilde{\chi}_0 = 1 + \iint_C \frac{\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}}{2\pi\mathrm{i}\lambda} \iint_C \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}\ \chi(\mu, \overline{\mu})\, R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})\, \mathrm{e}^{F(\mu)-F(\lambda)} \tag{28}$$

and

$$\chi_{-1} = -\iint_C \frac{\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}}{2\pi\mathrm{i}} \iint_C \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}\ \chi(\mu, \overline{\mu})\, R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})\, \mathrm{e}^{F(\mu)-F(\lambda)} \tag{29}$$

where $F(\lambda)$ is given by formula (11).

The conditions of reality U and of potentiality of the operator L_1 give some restrictions on the kernel R_0 of the $\overline{\partial}$-problem (8). In the Nizhnik case ($\sigma = 1$) of NVN equations (1) with real $\xi = x + y$, $\eta = x - y$ space variables and $\overline{\kappa_1} = \kappa_1, \overline{\kappa_2} = \kappa_2$ the condition of reality of U leads from (11), (23) and (29) in the limit of weak fields to the following restriction on the kernel R_0 of the $\overline{\partial}$-problem:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \overline{R_0(-\overline{\mu}, -\mu;\ -\overline{\lambda}, -\lambda)}. \tag{30}$$

To the Veselov–Novikov case ($\sigma = \mathrm{i}, \kappa_1 = \kappa_2 = \kappa = \overline{\kappa}$) of NVN equations (1) with $z = \xi = x + \mathrm{i}y, \overline{z} = \eta = x - \mathrm{i}y$ the condition of reality of U leads from (11), (23) and (29) in the limit of weak fields to another restriction on the kernel R_0 of $\overline{\partial}$-problem:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\epsilon}{|\mu|^2|\lambda|^2\overline{\mu}\overline{\lambda}}\ \overline{R_0\left(-\frac{\epsilon}{\overline{\lambda}}, -\frac{\epsilon}{\lambda}, -\frac{\epsilon}{\overline{\mu}}, -\frac{\epsilon}{\mu}\right)}. \tag{31}$$

The potentiality condition for the operator L_1 in (26) means $V_2 = 0$ or due to (21) $\widetilde{\chi}_0 =$ constant, say $\widetilde{\chi}_0 = 1$, and according to (28) has the form

$$\iint_C \frac{\mathrm{d}\lambda \wedge \mathrm{d}\overline{\lambda}}{\lambda} \iint_C \mathrm{d}\mu \wedge \mathrm{d}\overline{\mu}\ \chi(\mu, \overline{\mu})\, R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})\, \mathrm{e}^{F(\mu)-F(\lambda)} = 0. \tag{32}$$

Various choices for the kernel R of the $\overline{\partial}$-problem (8) satisfying restrictions (30)–(32) lead to various classes of exact solutions of integrable nonlinear NVN equations (1).

In conclusion of this section, let us obtain some useful general formulae for calculations of rational solutions of NVN equations (1). From such types of solutions leads for example the following delta-kernel $R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda})$ of the $\overline{\partial}$-problem:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} \sum_p \widetilde{A}_p \, \delta(\mu - \Lambda_p) \, \delta(\lambda - \Lambda_p). \tag{33}$$

Here for simplicity we choose $\widetilde{A}_p$ as some complex constants, $\delta(\mu - \Lambda_p)$ and $\delta(\lambda - \Lambda_p)$ are the complex δ-function. $\Lambda_1, \Lambda_2, \ldots$ is some set of isolated points distinct from the origin. The main problem in constructing rational solutions is the problem of the choice of the set of points Λ_p and constants $\widetilde{A}_p$ in order to satisfy the conditions of reality and potentiality. Using (33) in (28) and (29) one obtains for $\widetilde{\chi}_0$ and χ_{-1} the expressions

$$\widetilde{\chi}_0 = 1 + \sum_p \frac{\widetilde{A}_p}{\Lambda_p} \chi(\Lambda_p) \qquad \chi_{-1} = -\mathrm{i} \sum_p \widetilde{A}_p \, \chi(\Lambda_p). \tag{34}$$

For the quantities $\chi(\Lambda_p)$ from integral equation (27) follows a simple algebraic system of equations:

$$\sum_p A_{pq} \, \chi(\Lambda_q) = 1 \tag{35}$$

where matrix A_{pq} has the form

$$A_{pq} = \delta_{pq}(1 + \mathrm{i} \, \widetilde{A}_p F'(\Lambda_p)) + \frac{\mathrm{i} \, \widetilde{A}_q (1 - \delta_{pq})}{\Lambda_p - \Lambda_q}. \tag{36}$$

Using the fact $\partial A_{pq}/\partial \xi = -\widetilde{A}_p \delta_{pq}$ one obtains from (23) and (34) the simple determinantal formula for rational solutions of NVN equations (1):

$$U(\xi, \eta, t) = -\epsilon + \partial^2_{\xi\eta} \ln(\det A). \tag{37}$$

The condition of potentiality ($\widetilde{\chi}_0 = 1$) due to (34) for solutions with kernel R_0 (33) has the form

$$\sum_p \frac{\widetilde{A}_p}{\Lambda_p} \chi(\Lambda_p) = \sum_{p,q} \frac{\widetilde{A}_p}{\Lambda_p} A^{-1}_{pq} = 0. \tag{38}$$

3. Rational solutions of NVN equation at $\sigma = 1$

For the Nizhnik version ($\sigma = 1$) of NVN equations (1) with $\overline{\kappa_1} = \kappa_1, \overline{\kappa_2} = \kappa_2$ to the reality condition (30) satisfies, for example, following kernel R_0 of the $\overline{\partial}$-problem (8):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2} \sum_k [A_k \, \delta(\mu - \lambda_k) \, \delta(\lambda - \lambda_k) + \overline{A}_k \, \delta(\mu + \lambda_k) \, \delta(\lambda + \lambda_k)] \tag{39}$$

with N pairs of real ($\overline{\lambda_k} = \lambda_k$) points $(\lambda_k, -\lambda_k)$ arranged symmetrically near the origin on the real axis on the complex plane. A_κ are some complex constants which must be chosen appropriately in order to satisfy the condition of potentiality (32) or (38). It is convenient to rewrite the kernel R_0 in a more compact form of the type (33), where the sets $\widetilde{A}$ and Λ are introduced by the formulae

$$\widetilde{A} := (A_1, \overline{A_1}, \ldots, A_N, \overline{A_N}) \qquad \Lambda := (\lambda_1, -\lambda_1, \ldots, \lambda_N, -\lambda_N). \tag{40}$$

One can show that the condition of potentiality (38) satisfies for the following choice of A_k in (39):

$$\frac{1}{\overline{\overline{A_k}}} - \frac{1}{A_k} = \frac{\mathrm{i}}{\lambda_k}. \tag{41}$$

It follows from (41) that

$$\frac{1}{A_k} = a_k - \frac{\mathrm{i}}{2\lambda_k} \qquad \frac{1}{\overline{\overline{A_k}}} = a_k + \frac{\mathrm{i}}{2\lambda_k} \qquad \kappa = 1, \ldots, N \tag{42}$$

where a_k are arbitrary real constants. Equivalently, in terms of the sets $\tilde{A}$ and Λ, one has from (42):

$$\frac{1}{\widetilde{A}_p} = a_{[(p+1)/2]} - \frac{\mathrm{i}}{2\Lambda_p} \tag{43}$$

where $[(p+1)/2]$ denotes the entire part of $(p+1)/2$. The matrix (36) due to (11) and (43) has the form

$$A_{pq} = \left[\left(-\frac{\mathrm{i}}{2\Lambda_p} - X(\Lambda_p)\right)\delta_{pq} + \frac{\mathrm{i}(1-\delta_{pq})}{\Lambda_p - \Lambda_q}\right]\widetilde{A}_q \qquad (p, q = 1, \ldots, 2N) \tag{44}$$

where

$$X(\Lambda_p) = \xi + \frac{\epsilon}{\Lambda_p^2}\eta + 3\left(\kappa_1\Lambda_p^2 + \kappa_2\frac{\epsilon^3}{\Lambda_p^4}\right)t - a_{[(p+1)/2]} \tag{45}$$

and rational solutions corresponding to the kernel R_0 of the type (39) are given by formula (37) with matrix A_{pq} (44). All such solutions are evidently singular.

In the simplest case $N = 1$ $(\Lambda_1, \Lambda_2) = (\lambda_1, -\lambda_1)$, $X(\pm\lambda_1) = X(\lambda_1)$ it follows from (44)

$$A_{pq} = \begin{pmatrix} A_1\left(-\dfrac{\mathrm{i}}{2\lambda_1} - X(\lambda_1)\right) & \dfrac{\mathrm{i}\overline{A}_1}{2\lambda_1} \\ -\dfrac{\mathrm{i}A_1}{2\lambda_1} & \overline{A}_1\left(\dfrac{\mathrm{i}}{2\lambda_1} - X(\lambda_1)\right) \end{pmatrix}. \tag{46}$$

For the solution (37) one obtains in this case a very simple formula,

$$U = -\epsilon - \frac{2\epsilon/\lambda_1^2}{X(\lambda_1)^2}. \tag{47}$$

This solution has a singularity along the line $X(\lambda_1) = 0$. Such a line singularity, of second-order pole type, propagates in the plane (ξ, η) with speed

$$|\vec{V}| = [3\kappa_1\lambda_1^2 + 3\kappa_2\epsilon^3/\lambda_1^4]/\sqrt{1 + \epsilon^2/\lambda_1^4}.$$

The general solution (37) is the superposition of such simple solutions (47) with second-order pole-type singularities which interact with each other elastically.

Another simple delta-kernel R_0 of the $\overline{\partial}$-problem (8) satisfying the reality condition (30) has the form

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}[A_k\,\delta(\mu - \mathrm{i}\alpha_k)\,\delta(\lambda - \mathrm{i}\alpha_k) + B_k\,\delta(\mu + \mathrm{i}\alpha_k)\,\delta(\lambda + \mathrm{i}\alpha_k)] \tag{48}$$

with N pairs of pure imaginary points $(\mathrm{i}\alpha_k, -\mathrm{i}\alpha_k)$, $\overline{\alpha_k} = \alpha_k$, arranged symmetrically near the origin on the imaginary axis on the complex plane. Here $\overline{A_k} = A_k$, $\overline{B_k} = B_k$ are some real constants which must be chosen appropriately in order to satisfy the potentiality condition (32) or (38). It is convenient to rewrite the kernel R_0 (48) in a general compact form of the type (33), where the sets $\widetilde{A}$ and Λ are introduced by the formulae

$$\widetilde{A} := (A_1, B_1, \ldots, A_N, B_N) \qquad \Lambda := (\mathrm{i}\alpha_1, -\mathrm{i}\alpha_1, \ldots, \mathrm{i}\alpha_N, -\mathrm{i}\alpha_N). \tag{49}$$

One can show that in this case the potentiality condition (38) is satisfied for the following choice of A_k and B_k in (48):

$$\frac{1}{B_k} - \frac{1}{A_k} = \frac{1}{\alpha_k}. \tag{50}$$

It follows from (50) that $1/A_k = a_k - 1/(2\alpha_k)$, $1/B_k = a_k + 1/(2\alpha_k)$ with arbitrary real constants a_k, in terms of the sets $\widetilde{A}$ and Λ the last relations take the form:

$$\frac{1}{\widetilde{A}_p} = a_{[(p+1)/2]} - \frac{\mathrm{i}}{2\Lambda_p} \qquad p = 1, \ldots, 2N. \tag{51}$$

The rational solutions have the simple determinantal form (37) with the following matrix A:

$$A_{pq} = \left[\left(-\frac{\mathrm{i}}{2\Lambda_p} - X(\Lambda_p)\right)\delta_{pq} + \frac{\mathrm{i}(1-\delta_{pq})}{\Lambda_p - \Lambda_q}\right]A_q \qquad (p, q = 1, \ldots, 2N) \tag{52}$$

where

$$X(\Lambda_p) = \xi + \frac{\epsilon}{\Lambda_p^2}\eta + 3\left(\kappa_1\Lambda_p^2 + \kappa_2\frac{\epsilon^3}{\Lambda_p^4}\right)t - \widetilde{a}_p \tag{53}$$

and the constants $\widetilde{a}_p$ belong to the set $\widetilde{a} = (\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_{2N}) := (a_1, a_1, \ldots, a_N, a_N)$.

In the simplest case $N = 1$, $\Lambda = (\mathrm{i}\alpha_1, -\mathrm{i}\alpha_1)$ one obtains from (52),

$$A_{pq} = \begin{pmatrix} A_1\left(-\dfrac{1}{2\alpha_1} - X(\mathrm{i}\alpha_1)\right) & \dfrac{B_1}{2\alpha_1} \\ -\dfrac{A_1}{2\alpha_1} & B_1\left(\dfrac{1}{2\alpha_1} - X(\mathrm{i}\alpha_1)\right) \end{pmatrix} \tag{54}$$

and

$$U(\xi, \eta, t) = -\epsilon + \frac{2\epsilon/\alpha_1^2}{(\xi - (\epsilon/\alpha_1^2)\,\eta - 3(\kappa_1\alpha_1^2 - \kappa_2\epsilon^3/\alpha_1^4)t - a_1)^2}. \tag{55}$$

This solution also has second-order pole-type line singularity and the general formula (37) gives the superposition of such simple solutions as (55) which interact which each other elastically.

As a product our procedure allows one to also calculate the eigenfunctions—exact solutions of the linear auxiliary problems (2) and, in particular, of the classical Klein–Gordon equation (6). For the eigenfunctions satisfying this equation one can choose due to (25) the following expressions:

$$\psi(\xi, \eta, t) = \chi(\Lambda_p)\,\mathrm{e}^{F(\Lambda_p)} \qquad p = 1, \ldots, 2N \tag{56}$$

or

$$\psi(\xi, \eta, t) = \chi(\lambda)\,\mathrm{e}^{F(\lambda)} \qquad p = 1, \ldots, 2N \tag{57}$$

where $\chi(\lambda)$ satisfies equation (27) and $\chi(\Lambda_p)$ satisfies the system (35). As exact eigenfunctions of linear problems one can also choose arbitrary linear combinations of those defined in (56) and/or in (57). In the above two considered simplest cases (39) and (48) of the kernel R_0 with the eigenfunctions satisfying the Klein–Gordon equation (6), one can choose, for example, the form

$$\psi = \frac{\exp[\pm i(\lambda_1\xi - (\epsilon/\lambda_1)\eta + (\kappa_1\lambda_1^3 - \kappa_2\epsilon^3/\lambda_1^3)t)]}{\xi + (\epsilon/\lambda_1^2)\,\eta + 3(\kappa_1\lambda_1^2 + \kappa_2\epsilon^3/\lambda_1^4)t - a_1} \tag{58}$$

$$\psi = \frac{\exp[\pm(\alpha_1\xi + (\epsilon/\alpha_1)\eta - (\kappa_1\alpha_1^3 - \kappa_2\epsilon^3/\alpha_1^3)t)]}{\xi - (\epsilon/\alpha_1^2)\,\eta - 3(\kappa_1\alpha_1^2 - \kappa_2\epsilon^3/\alpha_1^4)t - a_1} \tag{59}$$

for the first (λ_1-real) and second ($i\alpha_1$-pure imaginary) cases, respectively.

In the above two studied examples (39) and (48) of kernel R_0, the corresponding solutions of equation (1) and the eigenfunctions ψ are singular. In order to construct non-singular rational solutions one must choose a more complicated set $\Lambda = (\Lambda_1, \Lambda_2, \ldots)$ of points of non-zero values of R_0 on the complex plane. The idea of choosing such a kernel is very simple. One starts with two terms $A_k\,\delta(\mu - \lambda_k)\,\delta(\lambda - \lambda_k) + B_k\,\delta(\mu + \lambda_k)\,\delta(\lambda + \lambda_k)$ in R_0 having non-zero values at two complex points λ_k and $-\lambda_k$. Then 'the continuation by reality condition' (30) gives another two terms $\overline{A}_k\,\delta(\mu + \overline{\lambda}_k)\,\delta(\lambda + \overline{\lambda}_k) + \overline{B}_k\,\delta(\mu - \overline{\lambda}_k)\,\delta(\lambda - \overline{\lambda}_k)$ of the kernel R_0. So let us consider a delta-kernel R_0, more general than (39) and (48) satisfying the reality condition (30):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}[A_k\,\delta(\mu - \lambda_k)\,\delta(\lambda - \lambda_k) + \overline{A}_k\,\delta(\mu + \overline{\lambda}_k)\,\delta(\lambda + \overline{\lambda}_k)$$
$$+B_k\,\delta(\mu + \lambda_k)\,\delta(\lambda + \lambda_k) + \overline{B}_k\,\delta(\mu - \overline{\lambda}_k)\,\delta(\lambda - \overline{\lambda}_k)] \tag{60}$$

with N quartets of complex points $(\lambda_k, -\overline{\lambda}_k, -\lambda_k, \overline{\lambda}_k)$ arranged symmetrically near the origin on the complex plane. Evidently (39) and (48) are two degenerate particular cases of (60). The constants A_k and B_k in (60) must be chosen appropriately in order to satisfy the potentiality condition (32) (or (38)). It is convenient to rewrite the kernel (60) in the general compact form of the type (33) with the sets $\widetilde{A}$ and Λ being given by the formulae

$$\begin{aligned}\widetilde{A} &:= (A_1, \overline{A}_1, B_1, \overline{B}_1; \cdots; A_N, \overline{A}_N, B_N, \overline{B}_N)\\ \Lambda &:= (\lambda_1, -\overline{\lambda}_1, -\lambda_1, \overline{\lambda}_1; \cdots; \lambda_N, -\overline{\lambda}_N, -\lambda_N, \overline{\lambda}_N).\end{aligned} \tag{61}$$

Some lengthy calculations show that the condition of potentiality (38) for the kernel (60) is satisfied for the following choice of the constants A_k and B_k in (60):

$$\frac{1}{B_k} - \frac{1}{A_k} = \frac{i}{\lambda_k} \tag{62}$$

It follows from (62) that $1/A_k = a_k - i/(2\lambda_k)$, $1/B_k = a_k + i/(2\lambda_k)$, with arbitrary complex constants a_k. Equivalently, in terms of the sets $\widetilde{A}$ and Λ defined in (61) and the set $\widetilde{a}$ defined by the formula

$$\widetilde{a} := (\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_{4N}) := (a_1, \overline{a}_1, a_1, \overline{a}_1; \cdots; a_N, \overline{a}_N, a_N, \overline{a}_N) \tag{63}$$

from (62) follow the relations

$$\frac{1}{\widetilde{A}_p} = \widetilde{a}_p - \frac{i}{2\Lambda_p}. \tag{64}$$

The rational solutions U corresponding to the kernel R_0 of the type (60) have the general determinantal form (37) with matrix A,

$$A_{pq} = \left[\left(-\frac{i}{2\Lambda_p} - X(\Lambda_p)\right)\delta_{pq} + \frac{i(1 - \delta_{pq})}{\Lambda_p - \Lambda_q}\right]\widetilde{A}_q \qquad (p, q = 1, \ldots, 4N) \tag{65}$$

where

$$X(\Lambda_p) = \xi + \frac{\epsilon}{\Lambda_p^2}\eta + 3\left(\kappa_1\Lambda_p^2 + \kappa_2\frac{\epsilon^3}{\Lambda_p^4}\right)t - \widetilde{a}_p. \tag{66}$$

In the simplest case $N = 1$ and $\Lambda = (\lambda_1, -\overline{\lambda}_1, -\lambda_1, \overline{\lambda}_1)$ the calculations using (65) and (66) lead to the following expression for $\det A$:

$$\det A = |A_1|^2|B_1|^2\left[|X(\lambda_1)|^2 + \frac{1}{4}\left(\frac{1}{\lambda_{1I}^2} - \frac{1}{\lambda_{1R}^2}\right)\right]^2. \tag{67}$$

Using (37) and (67) for the solution $U(\xi, \eta, t)$ one obtains the formula

$$U(\xi, \eta, t) = -\epsilon - 2\epsilon\frac{(\lambda_1 X(\lambda_1))^2 + (\overline{\lambda_1 X}(\lambda_1))^2 - 1/2(\lambda_{1I}^2 - \lambda_{1R}^2)^2/(\lambda_{1I}^2\lambda_{1R}^2)}{(|\lambda_1 X(\lambda_1)|^2 + (|\lambda_1|^2/4)(1/\lambda_{1I}^2 - 1/\lambda_{1R}^2))^2}. \tag{68}$$

The solution (68) is evidently non-singular for $|\lambda_{1I}| < |\lambda_{1R}|$ and represents a localized lump decreasing at infinity to $-\epsilon$ rationally (as $(\xi^2 + \eta^2)^{-1}$) and moving on the plane ξ, η with constant velocity.

4. Rational solutions of NVN equation at $\sigma = \mathrm{i}$

Quite analogously to the case $\sigma = 1$ of the previous section one can study the Veselov–Novikov version of the NVN equations (1) with $\sigma = \mathrm{i}$. For this case with the assumption $\kappa_1 = \overline{\kappa}_2 = \kappa$ to the reality condition (31) satisfies, for example, the following delta-kernel R_0 of the $\overline{\partial}$-problem (8):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}\left[A_k\,\delta(\mu - \lambda_k)\,\delta(\lambda - \lambda_k) + \frac{\epsilon^3\overline{A}_k}{|\mu|^2|\lambda|^2\overline{\mu}\overline{\lambda}}\,\delta\left(\frac{\epsilon}{\overline{\mu}} + \lambda_k\right)\delta\left(\frac{\epsilon}{\overline{\lambda}} + \lambda_k\right)\right]$$

$$= \frac{\pi}{2}\sum_{k=1}^{N}\left[A_k\,\delta(\mu - \lambda_k)\,\delta(\lambda - \lambda_k) + \frac{\epsilon\overline{A}_k}{\overline{\lambda}_k^2}\,\delta\left(\mu + \frac{\epsilon}{\overline{\lambda}_k}\right)\delta\left(\lambda + \frac{\epsilon}{\overline{\lambda}_k}\right)\right]. \tag{69}$$

Let us impose an additional restriction on the points λ_k:

$$\frac{\epsilon}{\overline{\lambda}_k} = \lambda_k \qquad \epsilon = |\lambda_k|^2 > 0. \tag{70}$$

For the kernel R_0 (69) then follows a simpler expression:

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}\left[A_k\,\delta(\mu - \lambda_k)\,\delta(\lambda - \lambda_k) + \frac{\overline{A}_k\lambda_k}{\overline{\lambda}_k}\,\delta(\mu + \lambda_k)\,\delta(\lambda + \lambda_k)\right] \tag{71}$$

with N pairs of complex points $(\lambda_k, -\lambda_k)$ arranged symmetrically near the origin on the complex plane. A_k in (71) are some complex constants which must be chosen appropriately in order to satisfy the potentiality condition (32) (or (38)). It is convenient to rewrite the kernel R_0 in a more compact form of the type (33) with the sets $\widetilde{A}$ and Λ introduced by the formulae

$$\begin{aligned}&\widetilde{A} := (A_1, \lambda_1\overline{A}_1/\overline{\lambda}_1,\ A_2, \lambda_2\overline{A}_2/\overline{\lambda}_2, \ldots, A_N, \lambda_N\overline{A}_N/\overline{\lambda}_N,)\\ &\Lambda := (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_N, -\lambda_N)\end{aligned} \tag{72}$$

and λ_k satisfies the condition (70). One can show that the condition of potentiality (32) for the kernel (71) is satisfied for the following choice of the constants A_k and $B_k := \lambda_k\overline{A}_k/\overline{\lambda}_k$ in (71):

$$\frac{1}{B_k} - \frac{1}{A_k} = \frac{\mathrm{i}}{\lambda_k}. \tag{73}$$

It follows from (73) that

$$\frac{1}{A_k} = a_k - \frac{\mathrm{i}}{2\lambda_k} \qquad \frac{1}{B_k} = a_k + \frac{\mathrm{i}}{2\lambda_k} \tag{74}$$

with complex constants a_k satisfying (due to the definition $B_k := \lambda_k \overline{A}_k/\overline{\lambda}_k$) the condition $a_k\lambda_k/\overline{\lambda}_k = \overline{a}_k$. Equivalently, in terms of the sets $\widetilde{A}$, Λ (72) and $\widetilde{a}$ defined by the formula

$$\widetilde{a} := (\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_{2N}) := (a_1, \overline{a}_1, a_2, \overline{a}_2; \cdots; a_N, \overline{a}_N) \tag{75}$$

the relations (74) can be rewritten in the form

$$\frac{1}{\widetilde{A}_p} = \widetilde{a}_p - \frac{\mathrm{i}}{2\Lambda_p} \qquad (p = 1, \ldots, 2N). \tag{76}$$

Matrix A_{pq} due to (11) and (76) has the form

$$A_{pq} = \left[\left(-\frac{\mathrm{i}}{2\Lambda_p} - X(\Lambda_p)\right)\delta_{pq} + \frac{\mathrm{i}(1-\delta_{pq})}{\Lambda_p - \Lambda_q}\right]\widetilde{A}_q \qquad (p, q = 1, \ldots, 2N) \tag{77}$$

where the quantities

$$\Lambda_p X(\Lambda_p) := z\Lambda_p + \frac{\epsilon}{\Lambda_p}\overline{z} + 3\left(\kappa\Lambda_p^3 + \overline{\kappa}\frac{\epsilon^3}{\Lambda_p^3}\right)t - \Lambda_p\widetilde{a}_p \tag{78}$$

due to the relations (70) and $a_k\lambda_k/\overline{\lambda}_k = \overline{a}_k$ are real. Rational solutions corresponding to the kernel R_0 of the type (71) are given by the general formula (37) with $\xi = z = x + \mathrm{i}y$, $\eta = \overline{z} = x - \mathrm{i}y$:

$$U(z, \overline{z}, t) = -\epsilon + \frac{\partial^2}{\partial z \partial \overline{z}} \ln \det A. \tag{79}$$

In the simplest case $N = 1$ and $(\Lambda_1, \Lambda_2) = (\lambda_1, -\lambda_1)$ one obtains from (77)

$$A_{pq} = \begin{pmatrix} A_1\left(-\frac{\mathrm{i}}{2\lambda_1} - X(\lambda_1)\right) & \frac{\mathrm{i}\overline{A}_1}{2\overline{\lambda}_1} \\ -\frac{\mathrm{i}A_1}{2\lambda_1} & \frac{\lambda_1\overline{A}_1}{\overline{\lambda}_1}\left(\frac{\mathrm{i}}{2\lambda_1} - X(\lambda_1)\right) \end{pmatrix} \qquad \det A = \frac{|A_1|^2\lambda_1}{\overline{\lambda}_1} X^2(\lambda_1) \tag{80}$$

where due to (70) and (78) $\lambda_1 X(\lambda_1) := \lambda_1 z + \overline{\lambda_1}\overline{z} + 3(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda}_1^3)t - a_1\lambda_1$. For the solution (79) one obtains in this case a very simple formula:

$$U(z, \overline{z}, t) = -\epsilon - \frac{2|\lambda_1|^2}{(\lambda_1 z + \overline{\lambda_1}\overline{z} + 3(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3)t - a_1\lambda_1)^2}. \tag{81}$$

The corresponding wavefunction $\psi(z, \overline{z}, t)$ of the 2D stationary Schrödinger equation (6) is

$$\psi_{z\overline{z}} + \widetilde{U}(z, \overline{z})\psi = \epsilon\psi \tag{82}$$

one can choose due to (25), for example, the form

$$\psi = \frac{\exp[\pm\mathrm{i}(\lambda_1 z - \overline{\lambda_1}\overline{z} + 3(\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3)t)]}{\lambda_1 z + \overline{\lambda_1}\overline{z} + 3(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3)t - a_1\lambda_1}. \tag{83}$$

The solution (81) and wavefunction (83) are singular with the singularity along the line $X(\lambda_1) = 0$ on the plane (x, y). Singularity in (83) is of second-order pole type and propagates in the plane (x, y) with some constant velocity. A more general solution (79) is the superposition of such simple line singular solutions and also is singular. The wavefunction (83) due to a real argument in the exponent has a non-oscillating character.

One can obtain another simple delta-kernel R_0 satisfying the reality condition (32) in the following way. One starts from the simple term $R_0 = A_k\,\delta(\mu-\lambda_k)\,\delta(\lambda-\lambda_k)$, 'the continuation by reality condition' (31) adds to it the following part:

$$\frac{\epsilon^3\overline{A}_k}{|\mu|^2|\lambda|^2\overline{\mu}\overline{\lambda}}\,\delta\left(-\frac{\epsilon}{\overline{\mu}}-\lambda_k\right)\delta\left(-\frac{\epsilon}{\overline{\lambda}}-\lambda_k\right)=\frac{\epsilon\overline{A}_k}{\overline{\lambda}_k^2}\,\delta\left(\mu+\frac{\epsilon}{\overline{\lambda}_k}\right)\delta\left(\lambda+\frac{\epsilon}{\overline{\lambda}_k}\right). \tag{84}$$

Then as opposed to (70) let us impose another additional restriction on the points λ_k:

$$\frac{\epsilon}{\overline{\lambda}_k} = -\lambda_k \qquad \epsilon = -|\lambda_k|^2 \leqslant 0. \tag{85}$$

Due to (85) one obtains from (84) the term $-(\overline{A}_k\lambda_k/\overline{\lambda}_k)\,\delta(\mu-\lambda_k)\,\delta(\lambda-\lambda_k)$ which reproduces the initial one if the condition $\overline{A}_k\lambda_k/\overline{\lambda}_k = -A_k$ is fulfilled. The above observation leads to the following (different from (71)) choice of the delta-kernel R_0 satisfying the reality condition (31):

$$R_0(\mu,\overline{\mu};\lambda,\overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}[A_k\,\delta(\mu-\lambda_k)\,\delta(\lambda-\lambda_k) + B_k\,\delta(\mu+\lambda_k)\,\delta(\lambda+\lambda_k)]. \tag{86}$$

This kernel has non-zero values at N pairs of complex points $(\lambda_k, -\lambda_k)$ arranged symmetrically near the origin of the complex plane. In (86) the conditions (85) for the points λ_k and the conditions

$$\overline{A}_k\lambda_k/\overline{\lambda}_k = -A_k \qquad \overline{B}_k\lambda_k/\overline{\lambda}_k = -B_k \tag{87}$$

for constants A_k, B_k must be fulfilled. It is convenient to rewrite the kernel R_0 in a more compact form of the type (33) with the sets $\widetilde{A}$ and Λ being introduced by the formulae

$$\begin{aligned}&\widetilde{A} = (A_1,\ldots,A_{2N}) := (A_1, B_1, A_2, B_2, \ldots, A_N, B_N)\\&\Lambda := (\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \ldots, \lambda_N, -\lambda_N).\end{aligned} \tag{88}$$

One can show that the condition of potentiality (38) is satisfied for the following further restrictions on constants A_k and B_k:

$$\frac{1}{B_k} - \frac{1}{A_k} = \frac{\mathrm{i}}{\lambda_k}. \tag{89}$$

From (89) one obtains

$$\frac{1}{A_k} = a_k - \frac{\mathrm{i}}{2\lambda_k} \qquad \frac{1}{B_k} = a_k + \frac{\mathrm{i}}{2\lambda_k} \tag{90}$$

where due to (87) the complex constants a_k satisfy the condition

$$\overline{a}_k\overline{\lambda}_k/\lambda_k = -a_k. \tag{91}$$

Equivalently, in terms of the sets $\widetilde{A}$, Λ (88) and $\widetilde{a}$ given by the formula

$$\widetilde{a} := (\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_{2N}) := (a_1, a_1, a_2, a_2; \cdots; a_N, a_N) \tag{92}$$

the relations (90) can be rewritten in the form

$$\frac{1}{\widetilde{A}_p} = \widetilde{a}_p - \frac{\mathrm{i}}{2\Lambda_p} \qquad (p = 1, \ldots, 2N). \tag{93}$$

Matrix A_{pq} due to (11) and (93) has the form

$$A_{pq} = \left[\left(-\frac{\mathrm{i}}{2\Lambda_p} - \mathrm{i}X(\Lambda_p)\right)\delta_{pq} + \frac{\mathrm{i}(1-\delta_{pq})}{\Lambda_p - \Lambda_q}\right]\widetilde{A}_q \qquad (p, q = 1, \ldots, 2N) \tag{94}$$

where the quantities

$$\Lambda_p X(\Lambda_p) := -\mathrm{i}\left(z\Lambda_p + \frac{\epsilon}{\Lambda_p}\overline{z} + 3\left(\kappa\Lambda_p^3 + \overline{\kappa}\frac{\epsilon^3}{\Lambda_p^3}\right)t\right) + \mathrm{i}\Lambda_p\widetilde{a}_p \tag{95}$$

due to (85) and (91) are real. Rational solutions corresponding to the kernel R_0 of the type (86) are given by the general formula (79) with the matrix A_{pq} (94).

In the simplest case $N = 1$ and $\Lambda = (\lambda_1, -\lambda_1)$ it follows from (94):

$$A_{pq} = \begin{pmatrix} A_1\left(-\dfrac{\mathrm{i}}{2\lambda_1} - \mathrm{i}\,X(\lambda_1)\right) & \dfrac{\mathrm{i}\,B_1}{2\lambda_1} \\ -\dfrac{\mathrm{i}\,A_1}{2\lambda_1} & B_1\left(\dfrac{\mathrm{i}}{2\lambda_1} - \mathrm{i}\,X(\lambda_1)\right) \end{pmatrix} \qquad \det A = -\frac{A_1B_1}{\lambda_1^2}\,\lambda_1^2X^2(\lambda_1) \tag{96}$$

where due to (87) the quantity A_1B_1/λ_1^2 is real and $\lambda_1X(\lambda_1) := -\mathrm{i}(\lambda_1 z - \overline{\lambda_1}\overline{z} + 3(\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda}_1^3)t - a_1\lambda_1)$. For the solution (79) one obtains in this case a very simple formula:

$$U(z, \overline{z}, t) = -\epsilon + \frac{|\lambda_1|^2}{(\lambda_1 z - \overline{\lambda_1}\overline{z} + 3(\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3)t - a_1\lambda_1)^2}. \tag{97}$$

The corresponding wavefunction $\psi(z, \overline{z}, t)$ of the 2D stationary Schrödinger equation (6)

$$\psi_{z\overline{z}} + \widetilde{U}(z, \overline{z})\psi = \epsilon\psi \tag{98}$$

one can choose (due to (25)) in the form

$$\psi = \frac{\mathrm{i}\exp(\pm\mathrm{i}(\lambda_1 z + \overline{\lambda_1}\overline{z} + 3(\kappa\lambda_1^3 + \overline{\kappa}\overline{\lambda_1}^3)t))}{\lambda_1 z - \overline{\lambda_1}\overline{z} + 3(\kappa\lambda_1^3 - \overline{\kappa}\overline{\lambda_1}^3)t - a_1\lambda_1}. \tag{99}$$

The solution (97) and wavefunction (99) are singular with the singularity along the line $X(\lambda_1) = 0$ on the plane (x, y). The singularity in (97) is of second-order pole type and propagates in the plane (x, y) with some constant velocity. A more general solution (79) is the superposition of such a simple line singular solutions and is also singular. The wavefunction (99) due to the pure imaginary argument of the exponent has an oscillating character.

In the above studied two examples (71) and (86) of kernel R_0 the corresponding solution of equation (1) and the wavefunctions ψ of the linear auxiliary problems are singular. In order to construct a non-singular rational solution one must choose a more complicated set $\Lambda = (\Lambda_1, \Lambda_2, \ldots)$ of points of non-zero values of R_0 on the complex plane. As in the previous section one starts with two terms $A_k\,\delta(\mu - \lambda_k)\,\delta(\lambda - \lambda_k) + B_k\,\delta(\mu + \lambda_k)\,\delta(\lambda + \lambda_k)$ in R_0 having non-zero values at two complex points $\lambda_k, -\lambda_k$. Then 'the continuation by reality condition' (31) gives another two terms $(\overline{A_k}\epsilon/\overline{\lambda}_k^2)\,\delta(\mu + \epsilon/\overline{\lambda}_k)\,\delta(\lambda + \epsilon/\overline{\lambda}_k) + \overline{B_k}\epsilon/\overline{\lambda}_k^2\,\delta(\mu - \epsilon/\overline{\lambda}_k)\,\delta(\lambda - \epsilon/\overline{\lambda}_k)$ of the kernel R_0 which are needed to satisfy the reality condition apart from the initial two. So let us consider a delta-kernel R_0 more general than (71) and (86) ,satisfying the reality condition (31):

$$R_0(\mu, \overline{\mu}; \lambda, \overline{\lambda}) = \frac{\pi}{2}\sum_{k=1}^{N}\left[A_k\,\delta(\mu - \lambda_k)\,\delta(\lambda - \lambda_k) + \frac{\overline{A}_k\epsilon}{\overline{\lambda}_k^2}\,\delta\left(\mu + \frac{\epsilon}{\overline{\lambda}_k}\right)\delta\left(\lambda + \frac{\epsilon}{\overline{\lambda}_k}\right)\right.$$
$$\left. + B_k\,\delta(\mu + \lambda_k)\,\delta(\lambda + \lambda_k) + \frac{\overline{B}_k\epsilon}{\overline{\lambda}_k^2}\,\delta\left(\mu - \frac{\epsilon}{\overline{\lambda}_k}\right)\delta\left(\lambda - \frac{\epsilon}{\overline{\lambda}_k}\right)\right] \tag{100}$$

which has non-zero values on the complex plane at N quartets of complex points

$$(\lambda_k, -\epsilon/\overline{\lambda}_k - \lambda_k, \epsilon/\overline{\lambda}_k)$$

arranged symmetrically near the origin on the complex plane and going to each other by inversion relative to the origin and/or to the circle of radius $\sqrt{|\epsilon|}$. It is convenient to rewrite the kernel (100) in a more compact form of the type (33) with the sets $\widetilde{A}$ end Λ introduced by the formulae:

$$\begin{aligned}\widetilde{A} &:= (A_1, \epsilon\overline{A}_1/\overline{\lambda}_1^2, B_1, \epsilon\overline{B}_1/\overline{\lambda}_1; \ \ldots; \ A_N, \epsilon\overline{A}_N/\overline{\lambda}_N^2, B_N, \epsilon\overline{B}_N/\overline{\lambda}_N) \\ \Lambda &:= (\lambda_1, -\epsilon/\overline{\lambda}_1, -\lambda_1, \epsilon/\overline{\lambda}_1; \cdots; \lambda_N, -\epsilon/\overline{\lambda}_N, -\lambda_N, \epsilon/\overline{\lambda}_N).\end{aligned} \tag{101}$$

Some lengthy calculations show that the condition of potentiality (38) for the kernel (100) is satisfied for the following choice of the constants A_k and B_k in (100):

$$\frac{1}{B_k} - \frac{1}{A_k} = \frac{\mathrm{i}}{\lambda_k} \qquad k = 1, \ldots, N. \tag{102}$$

It follows from (102) that

$$\frac{1}{A_k} = a_k - \frac{\mathrm{i}}{2\lambda_k} \qquad \frac{1}{B_k} = a_k + \frac{\mathrm{i}}{2\lambda_k} \tag{103}$$

with arbitrary complex constants a_k. In terms of the sets $\widetilde{A}$, Λ (101) and the set $\widetilde{a}$ defined by the formula

$$\widetilde{a} := (\widetilde{a}_1, \widetilde{a}_2, \ldots, \widetilde{a}_{4N}) := (a_1, \overline{a}_1\overline{\lambda}_1^2/\epsilon, a_1, \overline{a}_1\overline{\lambda}_1^2/\epsilon; \cdots; a_N, \overline{a}_N\overline{\lambda}_N^2/\epsilon, a_N, \overline{a}_N\overline{\lambda}_N^2/\epsilon) \tag{104}$$

the relations (103) can be rewritten in the form

$$\frac{1}{\widetilde{A}_p} = \widetilde{a}_p - \frac{\mathrm{i}}{2\Lambda_p} \qquad (p = 1, \ldots, 4N). \tag{105}$$

The rational solutions U corresponding to the kernel R_0 (100) have the general determinantal form (79):

$$U(z, \overline{z}, t) = -\epsilon + \frac{\partial^2}{\partial z \partial \overline{z}} (\ln \det A) \tag{106}$$

with matrix A

$$A_{pq} = \left[\left(-\frac{\mathrm{i}}{2\Lambda_p} - X(\Lambda_p)\right)\delta_{pq} + \frac{\mathrm{i}(1 - \delta_{pq})}{\Lambda_p - \Lambda_q}\right]\widetilde{A}_q \qquad (p, q = 1, \ldots, 4N) \tag{107}$$

where

$$X(\Lambda_p) := z + \frac{\epsilon}{\Lambda_p^2}\overline{z} + 3\left(\kappa\Lambda_p^2 + \overline{\kappa}\frac{\epsilon^3}{\Lambda_p^4}\right)t - \widetilde{a}_p. \tag{108}$$

In the simplest case of one quartet of complex points $(\lambda_1, -\epsilon/\overline{\lambda}_1, -\lambda_1, \epsilon/\overline{\lambda}_1)$ one has from (106) the following expression for $\det A$:

$$\det A = |A_1|^2|B_1|^2 \left(|X(\lambda_1)|^2 - \frac{2\epsilon(\epsilon^2 + |\lambda_1|^4)}{(\epsilon^2 - |\lambda_1|^4)^2}\right)^2. \tag{109}$$

Using (79) and (108) for the solution $U(z, \overline{z}, t)$ one obtains the formula

$$U(z, \overline{z}, t) = -\epsilon - 2\epsilon\frac{\lambda_1^2 X(\lambda_1)^2 + \overline{\lambda}_1^2\overline{X}(\lambda_1)^2 + 2[(\epsilon^2 + |\lambda_1|^4)^2/(\epsilon^2 - |\lambda_1|^4)^2]}{\left(|\lambda_1 X(\lambda_1)|^2 - [2\epsilon\,|\lambda_1|^2(\epsilon^2 + |\lambda_1|^4)/(\epsilon^2 - |\lambda_1|^4)^2]\right)^2}. \tag{110}$$

This solution evidently is non-singular for $\epsilon < 0$ and represents a localized lump decreasing at infinity to $-\epsilon$ rationally (as $|z|^{-2}$) and moving on the plane x, y with constant velocity.

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